Linear partial differential equations of high order with constant coefficients

P. Sam Johnson

March 5, 2020

Overview

We are concerned in the course with partial differential equations with one dependent variable z and two independent variables x and y.

We discuss few methods to solve linear differential equations of n^{th} order with constant coefficients in three lectures.

Lagrange linear partial differential equations

The equation of the form

$$Pp + Qq = R$$

is known as **Lagrange linear equation** and P, Q and R are functions of y and z. To solve this type of equations it is enough to solve the equation which the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

From the above subsidiary equation we can obtain two independent solutions $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, then the solution of the Lagrange's equation is given by $\phi(u, v) = 0$.

There are two methods of solving the subsidiary equation known as **method of grouping** and **method of multipliers**.

Method of Grouping

Consider the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Take any two ratios of the above equation say the first two or first and third or second and third. Suppose we take $\frac{dx}{P} = \frac{dy}{Q}$ and if the functions P and Q may contain the variable z, then eliminate the variable z. Then the direct integration gives $u(x, y) = c_1, v(y, z) = c_2$, then the solution of the Lagrange's equation is given by $\phi(u, v) = 0$.

Method of multipliers

Choose any three multipliers ℓ, m, n which may be constants or functions of x, y and z such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\ell dx + mdy + ndz}{\ell P + mQ + nR}.$$

If the relation $\ell P + mQ + nR = 0$, then $\ell dx + mdy + ndz$. Now direct integration gives us a solution

$$u(x, y, z) = c_1.$$

Similarly any other set of multipliers ℓ', m', n' gives another solution

$$v(x, y, z) = c_2.$$

Examples on method of Grouping

Example 1.

Solve $xp + yq = z$.

Solution. *The subsidiary equation is* $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$. *Taking the first ratio we have* $\frac{dx}{x} = \frac{dy}{y}$. *Integrating we get*

$$\log\ x = \log\ y + \log c_1$$
$$\log \frac{x}{y} = \log\ c_1$$
$$\frac{x}{y} = c_1.$$

Taking the second and third ratios we have $\frac{dy}{y} = \frac{dz}{z}$. *Integrating we get*

$$\log y = \log\ z + \log\ c_2$$
$$\log \frac{y}{z} = \log c_2$$
$$\frac{y}{z} = c_2.$$

The required solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

Example 2.

Solve $xp + yq = x$*.*

Solution. *The subsidiary equation is* $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$*. Taking the first ratio we have* $\frac{dx}{x} = \frac{dy}{y}$*.*
Integrating we get

$$\log\ x = \log\ y + \log\ c_1$$
$$\frac{x}{y} = c_1.$$

Taking the first and third ratios we have

$$\frac{dx}{x} = \frac{dz}{x}$$
$$dx = dz.$$

Integrating we get

$$x = z + c_2$$
$$x - z = c_2.$$

The required solution is $\phi\left(\frac{x}{y}, x - z\right) = 0$*.*

Example 3.

Solve $\tan x p + \tan y q = \tan z$.

Solution. *The subsidiary equation is* $\frac{dx}{\tan\ x} = \frac{dy}{\tan\ y} = \frac{dz}{\tan\ z}$.

Integrating $\frac{dx}{\tan\ x} = \frac{dy}{\tan\ y}$ *we get*

$$\log \sin x = \log \sin y + \log c_1 \implies \log \frac{\sin x}{\sin y} = \log c_1 \qquad \implies \frac{\sin x}{\sin y} = c_1$$

Integrating $\frac{dy}{\tan y} = \frac{dz}{\tan z}$ *we get*

$$\log \sin y = \log \sin y + \log c_2 \implies \log \frac{\sin\ y}{\sin z} = \log c_2 \qquad \implies \frac{\sin y}{\sin z} = c_2.$$

The required solution is $\phi\left(\frac{\sin\ x}{\sin\ y}, \frac{\sin y}{\sin\ z}\right) = 0$.

Example 4.

Find the complete integral of the partial differential equation $(1-x)p + (2-y)q = 3-z$.
Solution. *The subsidiary equation is*

$$\frac{dx}{1-x} = \frac{dy}{2-y} = \frac{dz}{3-z}.$$

Integrating $\frac{dx}{1-x} = \frac{dy}{2-y}$ *we get*

$$-\log(1-x) = -\log(2-y) + \log c_1 \implies \frac{2-y}{1-x} = c_1.$$

Integrating $\frac{dx}{1-x} = \frac{dz}{3-z}$ *we get*

$$-\log(1-x) = -\log(3-z) + \log c_2 \implies \frac{3-z}{1-x} = c_2.$$

The requird solution is $\phi\left(\frac{2-y}{1-x}, \frac{3-z}{1-x}\right) = 0$.

Examples based on method of multipliers

Example 5.

Solve $(y-z)p+(z-x)q=(x-y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}.$$

Using the multipliers $1,1,1$ *we have*

$$\text{Each ratio } =\frac{dx+dy+dz}{y-z+z-x+x-y}=\frac{dx+dy+dz}{0}\implies x+y+z=c_1.$$

Using the multipliers x,y,z *we have*

$$\text{Each ratio } =\frac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)}=\frac{xdx+ydy+zdz}{0}\implies x^2+y^2+z^2=2c_2.$$

Hence the solution is $\phi(x+y+z,x^2+y^2+z^2)=0$.

Example 6.

Solve $x(y-z)p + y(z-x)q = z(x-y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}.$$

Using the multipliers $1, 1, 1$ *we have*

$$\textit{Each ratio} = \frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz} = \frac{dx+dy+dz}{0} \implies x+y+z = c_1.$$

Using the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\textit{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z+z-x+x-y)} \implies \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \implies xyz = c_2.$$

Hence the solution is $\phi(x+y+z, xyz) = 0$.

Example 7.

Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}.$$

Using the multipliers x, y, z *we have*

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + ydz}{0}$$
$$\Longrightarrow x^2 + y^2 + z^2 = c_1.$$

Choosing the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \Longrightarrow xyz = c_2.$$

The required solution is $\phi(x^2 + y^2 + z^2, xyz) = 0$.

Example 8.

Solve $x^2(y-z)+y^2(z-x)q=z^2(x-y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}.$$

Using the multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ *we have*

$$\text{Each ratio}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}\implies xyz=c_1.$$

Using the multipliers $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ *we have*

$$\text{Each ratio}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}\implies\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_2.$$

The required solution is $\phi(xyz,\frac{1}{x}+\frac{1}{y}+\frac{1}{z})=0$.

Example 9.

Solve $(4y-3z)p+(2z-4x)q=(3x-2y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{4y-3z}=\frac{dy}{2z-4x}=\frac{dz}{3z-2y}$. *Using the multipliers* $2,3,4$ *we have*

$$\text{Each ratio} = \frac{2dx+3dy+4dz}{2(4y-3z)+3(2z-4x)+4(3x-2y)} = \frac{2dx+3dy+4dz}{0}$$

$$\Rightarrow \quad 2dx+3dy+4dz=0 \implies 2x+3y+4z=0.$$

Using the multipliers x,y,z *we have*

$$\text{Each ratio} = \frac{xdx+ydy+zdz}{x(4y-3z)+y(2z-4x)+z(3x-2y)} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \quad xdx+ydy+zdz=0 \implies x^2+y^2+z^2=c_2.$$

The required solution $\phi(2x+3y+4z, x^2+y^2+z^2)=0$.

Example 10.

Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{x(y^2+z)} = \frac{dx}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$. *Using the multipliers* $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we have*

$$\text{Each ratio } = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 + z - x^2 - z + z^2 - y^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \implies \log\ x + \log\ y + \log\ z = \log\ c_1 \implies xyz = c_1.$$

Using the multipliers $x, y, -1$ *we have*

$$\text{Each ratio} = \frac{xdx + ydy - dz}{z^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{xdx + ydy - dz}{x^2y^2 + x^2z - y^2x^2 - y^2z - zx^2 + zy^2}$$

$$= \frac{xdx + ydy - dz}{0} \Rightarrow \quad xdx + ydy - dz = 0 \implies x^2 + y^2 - 2z = c_2.$$

The required solution is $\phi(xyz, x^2+y^2-2z) = 0$.

Example 11.

Find the general solution of $z(x-y) = x^2p - y^2q$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$. *Taking the first two ratios*

$$\frac{dx}{x^2} = \frac{dy}{-y^2} \implies -\frac{1}{x} = \frac{1}{y} + c_1 \implies \frac{1}{y} - \frac{1}{x} = c_1.$$

Adding first two ratios and comparing this with third

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{z(x-y)} \implies \frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{z(x-y)} \implies \frac{dx+dy}{x+y} = \frac{dz}{z}$$

$$\log(x+y) = \log z + \log c_2 \implies \log\frac{(x+y)}{z} = \log c_2 \implies \frac{x+y}{z} = c_2.$$

The required solution is $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{z+y}{z}\right) = 0$.

Example 12.

Solve $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is* $\frac{dx}{(x^2-y^2-z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$. *Taking the second and third ratios*

$$\frac{dy}{2xy} = \frac{dz}{2xz} \implies \frac{dy}{y} = \frac{dz}{z} \implies \log\ y = \log\ z + \log\ c_1 \implies \frac{y}{z} = c_1.$$

Using the multipliers x, y, z *we have*

$$\text{Each ratio} = \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}.$$

Comparing this with the second ratio

$$\frac{dy}{2xy} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \implies \frac{dy}{y} = \frac{2(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)}$$

$$\log\ y = \log(x^2 + y^2 + z^2) + \log c_2 \implies \frac{y}{x^2 + y^2 + z^2} = c_2.$$

Hence the solution is $\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0$.

Example 13.

Solve $(x^2 - yz)p + (y^2 - xz)q = z^2 - xy$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - xz} = \frac{dz}{z^2xy}.$$

Using the multipliers $1, 1, 1$ *we have*

$$\textit{Each ratio } = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - xz - xy}. \tag{1}$$

Using the multipliers x, y, z *we have*

$$\textit{Each ratio } = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}. \tag{2}$$

Solution (contd...)

Comparing (1) and (2) we have

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xz-xy} = \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$\frac{dx+dy+dz}{x^2+y^2+z^2-yz-xz-xy} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-xz-xy)}$$

$$dx+dy+dz = \frac{xdx+ydy+zdz}{(x+y+z)} \quad \Longrightarrow \quad xy+yz+xz = c_1.$$

Taking the first two ratios

$$\text{Each ratio } = \frac{dx-dy}{x^2-yz-(y^2-xz)} = \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dx-dy}{(x-y)(x+y+z)}. \tag{3}$$

Taking the second and third ratios

$$\text{Each ratio } = \frac{dy-dz}{y^2-xz-(z^2-xy)} = \frac{dy-dz}{y^2-z^2+x(y-z)} = \frac{dy-dz}{(y-z)(x+y+z)} \tag{4}$$

Comparing (3) and (4) we have

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} \quad \Longrightarrow \quad \frac{x-y}{y-z} = c_2.$$

Hence the solution is $\phi\left(xy+yz+xz, \frac{x-y}{y-z}\right) = 0$.

Example 14.

Solve $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x + y)}.$$

Using the multipliers $1, -1, -1$ *we have*

$$\text{Each ratio} = \frac{dx - dy - dz}{x^2 + y^2 + yz - x^2 - y^2 + xz - zx - xy} = \frac{dx - dy - dz}{0} \implies x - y - z = c_1.$$

Using the multipliers $x, y, 0$ *we have*

$$\text{Each ratio} = \frac{xdx + ydy}{x^3 + xy^2 + xyz + x^2y + y^3 - xyz} = \frac{dz}{z(x + y)}$$

$$\frac{xdx + ydy}{(x + y)(x^2 + y^2)} = \frac{dz}{z(x + y)} \implies \frac{xdx + ydy}{x^2 + y^2} = \frac{dz}{z} \implies \frac{x^2 + y^2}{z^2} = c_2.$$

Hence the solution is $\phi\left(x - y - z, \frac{x^2+y^2}{z^2}\right) = 0$.

Example 15.

Solve $(x+y)zp+(x-y)zq = x^2+y^2$.

Solution. *The given equation is Lagrange equation. Hence the subsidiary equation is*

$$\frac{dx}{(x+y)z} = \frac{dy}{(x-y)z} = \frac{dz}{x^2+y^2}.$$

Using the multipliers $x, -y, -z$ *we have*

$$\text{Each ratio } = \frac{xdx - ydy - zdz}{x^2z + xyz - xyz + y^2z - x^2z - y^2z} = \frac{xdx - ydy - zdz}{0}$$
$$\Rightarrow \quad xdx - ydy - zdz = 0 \implies x^2 - y^2 - z^2 = c_1.$$

Using the multipliers $y, x, -z$ *we have*

$$\text{Each ratio } = \frac{ydx + xdx - zdz}{xyz + y^2z + xz^2 - xyz - xz^2 - y^2z} = \frac{ydx + xdy - zdz}{0}$$
$$\implies ydx + xdx - zdz = 0 \implies 2xy - z^2 = c_2.$$

Hence the solution is $\phi(x^2-y^2, z^2, 2xy - z^2) = 0$.

Linear partial differential equations of high order with constant coefficients

A linear differential equation of n^{th} order with constant coefficients of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} +$$
$$b_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + b_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + b_2 \frac{\partial^{n-1} z}{\partial x^{n-3}\partial y^2} + \cdots + b_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}$$
$$+ \cdots + \ell_0 \frac{\partial^2 z}{\partial x^2} + \ell_1 \frac{\partial^2 z}{\partial x \partial y} + \ell_2 \frac{\partial^2 z}{\partial y^2} + \ell_3 \frac{\partial z}{\partial x} + \ell_4 \frac{\partial z}{\partial y} + \ell_5 z = G(x, y)$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_{n-1}, \ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ are constants.

Homogeneous linear partial differential equations

Using the standard notation $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$ the above equation can be written as

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n +$$
$$b_0 D^{n-1} + b_1 D^{n-2} D' + b_2 D^{n-3} D'^2 + \cdots + b_{n-1} D'^{n-1} +$$
$$+ \cdots + \ell_0 D^2 + \ell_1 DD' + \ell_2 D'^2 + \ell_3 D + \ell_4 D' + \ell_5]z = G(x,y).$$

The **homogenous equations of order** n is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} + = G(x,y)$$
$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 \cdots + a_n D'^n]z = G(x,y).$$

Complementary functions

To find the complementary functions for the linear homogenous partial differential equation of order n we consider

$$[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n]z = 0. \tag{3}$$

Let us assume that

$$z = f(y + mx)$$

be a solution of the above equation. Differentiating partially with respect to x we get

$$\begin{aligned} Dz &= mf'(y + mx) \\ D^2 z &= m^2 f''(y + mx) \\ &\vdots \\ D^n z &= m^n f^{(n)}(y + mx). \end{aligned}$$

Complementary functions

Similarly differentiating partially with respect to y we get $D'^n z = f^{(n)}(y + mx)$. And the mixed partial derivative is given by

$$D^{n-r} D'^r z = m^{n-r} f^{(n)}(y + mx).$$

Substituting these values in (3) we get

$$[a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n] \ \ f^{(n)}(y + mx) = 0.$$

Since f is arbitrary $f^{(n)}(y + mx) \neq 0$. Hence

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0. \tag{4}$$

This equation is known as **auxiliary equation** which is an algebraic equation of n^{th} degree in m hence by fundamental theorem of algebra it has n roots.

Complementary functions

Case (i) : If the roots are distinct (real or complex) say $m_1, m_2, \ldots, m_n$, then the complementary function is given by

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots + f_n(y + m_n x).$$

Case (ii) : If the r roots are equal say $m_1 = m_2 = \cdots = m_r$, then the complementary function is given by

$$z = f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \cdots + x^r f_r(y + m_1 x) \\ + f_{r+1}(y + m_{r+1} x) + \cdots + f_n(y + m_n x).$$

For $r = 2$ we have

$$z = f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x).$$

For $r = 3$ we have

$$z = f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + f_4(y + m_4 x) + \cdots + f_n(y + m_n x).$$

Example 16.

Solve $(D^2 - 5DD' + 6D'^2)z = 0.$

Solution.

$$\begin{aligned} \textit{The auxillary equation is } m^2 - 5m + 6 &= 0 \\ (m-2)(m-3) &= 0 \\ m &= 2, 3. \end{aligned}$$

$$z = f_1(y + 2x) + f_2(y + 3x).$$

Example 17.

Solve $(D^2 - 4DD' + 4D'^2)z = 0.$

Solution.

$$\begin{aligned} \textit{The auxillary equation is } m^2 - 4m + 4 &= 0 \\ (m-z)^2 &= 0 \\ m &= 2, 2. \end{aligned}$$

$$z = f_1(y + 2x) + xf_2(y + 2x).$$

Examples

Example 18.

Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0.$

Solution.

$$\begin{aligned} \text{The auxillary equation is } m^3 - 6m^2 + 11m - 6 &= 0 \\ (m-1)(m-2)(m-3) &= 0 \\ m &= 1, 2, 3. \end{aligned}$$

$$z = f_1(y+x) + f_2(y+2x) + f_2(y+2x).$$

Example 19.

Solve $(D^4 - 16D'^4)z = 0.$

Solution.

$$\begin{aligned} \text{The auxillary equation is } m^4 - 16 &= 0 \\ (m^2-4)(m^2+4) &= 0 \\ m &= \pm 2, \pm 2i. \end{aligned}$$

$$z = f_1(y+2x) + f_2(y-2x) + f_3(y+2ix) + f_4(y-2ix).$$

Example 20.

Solve $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0.$

Solution.

$$\begin{aligned} \textit{The auxillary equation is} \quad m^4 - 2m^3 + 2m - 1 &= 0 \\ (m^2-1)(m-1)^2 &= 0 \\ (m+1)(m-1)^3 &= 0 \\ m &= -1, 1, 1, 1. \end{aligned}$$

$$z = f_1(y-x) + f_2(y+x) + xf_3(y+x) + x^2 f_4(y+x).$$

The particular Integral

Let $F(D, D')z = G(x, y)$ be homogeneous of non-homogeneous linear partial differential equation with constant coefficients. Then the particular integral **(P.I.)** is given by

$$P.I. = \frac{1}{F(D, D')} G(x, y).$$

Case (i). If $G(x, y) = e^{ax+by}$ then the particular integral is given by

$$P.I. = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$$

provided $F(a, b) \neq 0$.

The particular Integral

If $F(a,b) = 0, (D - \frac{a}{b}D')$ or its power will be a factor for $F(D,D') = 0$. In this case it can be factorized and proceed as follows:

$$P.I. = \frac{1}{(D - \frac{a}{b}D')F_1(D,D')} e^{ax+by} = \frac{1}{F_1(a,b)} x \; e^{ax+by}$$

provided $F_1(a,b) \neq 0$.

$$P.I. = \frac{1}{(D - \frac{a}{b}D')^2 F_2(D,D')} e^{ax+by} = \frac{1}{F_2(a,b)} \frac{x^2}{2} e^{ax+by}$$

provided $F_2(a,b) \neq 0$.

$$\vdots$$

$$P.I. = \frac{1}{(D - \frac{a}{b}D')^r F_r(D,D')} e^{ax+by} = \frac{1}{F_r(a,b)} \frac{x^r}{r!} e^{ax+by}$$

provided $F_r(a,b) \neq 0$.

Example 21.

Solve $(D^2 - 4DD' + 3D'^2)z = e^{2x+3y}$.

Solution.

$$\text{The auxillay equation is } m^2 - 4m + 3 = 0$$
$$(m-1)(m-3) = 0$$
$$m = 1, 3.$$
$$C.F = f_1(y+x) + f_2(y+3x)$$
$$P.I = \frac{1}{D^2 - 4DD' + 3D'^2} e^{2x+3y}$$
$$= \frac{1}{2^2 - 4(2)(3) + 3(3)^2)} e^{2x+3y}$$
$$= \frac{1}{4 - 24 - 27} e^{2x+3y}$$
$$= \frac{1}{7} e^{2x+3y}.$$
$$z = f_1(y+x) + f_2(y+3x) + \frac{1}{7} e^{2x+3y}.$$

Example 22.

Solve $(D^2 - D'^2)z = e^{x-y}$.
Solution.

$$\begin{aligned} \text{The auxillary equation is } m^2 - 1 &= 0 \\ (m-1)(m+1) &= 0 \\ m &= \pm 1. \end{aligned}$$

$$C.F = f_1(y+x) + f_2(y-x).$$

$$\begin{aligned} P.I. &= \frac{1}{D^2 - D'^2} e^{x-y} \\ &= \frac{1}{(D-D')(D+D')} e^{x-y} \\ &= \frac{1}{(1-(-1))(D+D')} e^{x-y} \\ &= \frac{1}{2}\, x\, e^{x-y}. \end{aligned}$$

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{2}\, x\, e^{x-y}.$$

Example 23.

Solve $(D^2 - 4DD' + 4D'^2) = e^{2x+y}$.
Solution.

$$\begin{aligned} \text{The auxillary equation is } m^2 - 4m + 4 &= 0 \\ (m-2)^2 &= 0 \\ m &= 2, 2. \\ C.F = f_1(y+2x) + xf_2(y+2x). \end{aligned}$$

$$\begin{aligned} P.I &= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y} \\ &= \frac{1}{(D-2D')^2} e^{2x+y} \\ &= \frac{x^2}{2} e^{2x+y}. \end{aligned}$$

$$z = f_1(y+2x) + xf_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Example 24.

Solve $(D^3 - 5D^2D' + 8DD'^2 - 4D'^3)z = e^{2x+y}$.

Solution.

$$\text{The auxillary equation is } m^3 - 5m^2 + 8m - 4 = 0$$
$$(m-1)(m-2)(m-2) = 0$$
$$m = 1, 2, 2.$$

$$C.F = f_1(y+x) + f_2(y+2x) + xf_2(y+2x).$$

$$P.I. = \frac{1}{D^3 - 5D^2D' + 8DD'^2 - 4D'^3} e^{2x+y}$$
$$= \frac{1}{(D-D')(D-2D')^2} E^{2x+y}$$
$$= \frac{x^2}{2} e^{2x+y}.$$

$$z = f_1(y+x) + f_2(y+2x) + xf_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Case (ii)

If $G(x, y) = \cos(ax + by)$ or $\sin(ax + by)$ then the particular integral is given by

$$P.I. = \frac{1}{F(D, D')} \cos(ax + by) \ (OR) \ \sin(ax + by)$$
$$= R.P. \ or \ I.P. \frac{1}{F(D, D')} e^{i(ax+by)},$$

then proceed as in the **Case (i)**.

Example 25.

Solve $(D^2 - DD' - 2D'^2)z = \sin(3x + 4y)$.

Solution.

$$\begin{aligned}
\text{The auxiliary equation is } m^2 - m - 2 &= 0 \\
(m-2)(m+1) &= 0 \\
m &= 2, -1. \\
C.F = f_1(y+2x) + f_2(y-x). & \\
P.I. &= \frac{1}{D^2 - DD' - 2D'^2} \sin(3x+4y) \\
&= I.P. \frac{1}{D^2 - DD' - 2D'^2} e^{i(3x+4y)} \\
&= I.P. \frac{1}{(3i)^2 - (3i)(4i) - 2(4i)^2} e^{i(3x+4y)} \\
&= I.P. \frac{1}{-9 + 12 + 32} e^{i(3x+4y)} \\
&= I.P. \frac{1}{35} [\cos(3x+4y) + i \ \sin(3x+4y)] \\
&= \frac{1}{35} \quad \sin(3x+4y).
\end{aligned}$$

$$z = f_1(y+2x) + f_2(y-x) + \frac{1}{35} \sin(3x+4y).$$

Example 26.

Solve $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$.
Solution.

$$
\begin{aligned}
\textit{The auxiliary equation is } m^2 - 2m + 1 &= 0\\
(m-1)^2 &= 0\\
m &= 1, 1.
\end{aligned}
$$

$$C.F = f_1(y+x) + xf_2(y+x).$$

$$
\begin{aligned}
P.I &= \frac{1}{D^2 - 2DD' + D'^2}\cos(x-3y)\\
&= R.P.\frac{1}{D^2 - 2DD' + D'^2}e^{i(x-3y)}\\
&= R.P.\frac{1}{(i)^2 - 2(i)(-3i) + (-3i)^2}e^{i(x-3y)}\\
&= R.P.\frac{1}{-1-6-9}e^{i(x-3y)}\\
&= R.P.\frac{1}{-16}[\cos(x-3y) + i\ \sin(x-3y)]\\
&= -\frac{1}{16}\cos(x-3y).
\end{aligned}
$$

$$z = f_1(y+x) + xf_2(y+x) - \frac{1}{16}\cos(x-3y).$$

Example 27.

Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.
Solution.

$$\text{The auxiliary equation is } m^2 + 4m - 5 = 0$$
$$(m-1)(m+5) = 0$$
$$m = 1, -5.$$
$$C.F = f_1(y + x) + f_2(y - 5x).$$

$$\begin{aligned} P.I &= \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y) \\ &= I.P. \frac{1}{D^2 + 4DD' - 5D'^2} e^{i(2x+3y)} \\ &= I.P. \frac{1}{(2i)^2 + 4(2i)(3i) - 5(3i)^2} e^{i(2x+3y)} \\ &= I.P. \frac{1}{-4 - 24 + 45} e^{i(2x+3y)} \\ &= I.P. \frac{1}{17} [\cos(2x + 3y) + i \ \sin(2x + 3y)] \\ &= \frac{1}{17} \sin(2x + 3y). \end{aligned}$$

$$z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17} \sin(2x + 3y).$$

Example 28.

Solve $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$.

Solution.

$$\begin{aligned} \text{The auxiliary equation is } 2m^2 - 5m + 2 &= 0 \\ (2m-1)(m-2) &= 0 \\ m &= 2, \frac{1}{2}. \end{aligned}$$

$$C.F. = f_1(y+2x) + f_2(y + \frac{1}{2}x).$$

$$\begin{aligned} P.I. &= \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \ \sin(2x+y) \\ &= I.P. \frac{1}{(2D - D')(D - 2D')} 5e^{i(2x+y)} \\ &= I.P. \frac{1}{(2(2i) - i)} 5x \ e^{i(2x+y)} \\ &= I.P. \frac{-i}{3} 5x[\cos(2x+y) + i \ \sin(2x+y)] \\ &= -\frac{5}{3}x \ \cos(2x+y). \end{aligned}$$

$$z = f_1(y+2x) + f_2(y + \frac{1}{2}x) - \frac{5}{3}x \ \cos(2x+y).$$

Example 29.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos(2y)$.

Solution.

$$\begin{aligned} \text{The auxillary equation is } m^3 + m^2 - m - 1 &= 0 \\ m^2(m+1) - (m+1) &= 0 \\ (m^2 - 1)(m+1) &= 0 \\ m &= 1, -1, -1. \end{aligned}$$

$$C.F = f_1(y+x) + f_2(y-x) + xf_3(y-x).$$

$$\begin{aligned} P.I. &= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x \cos(2y) = R.P \frac{1}{D^3 + D^2D' - DD'^2 - D'^3} e^x e^{i2y} \\ &= R.P \frac{1}{(1)^3 + (1)^2(2i) - (1)(2i)^2 - (2i)^3} e^{x+i2y} = R.P. \frac{1}{1 + 2i + 4 + 8i} e^{x+i2y} \\ &= R.P. \frac{1}{5(1+2i)} e^{x+i2y} = R.P. \frac{1}{5(1+2i)} \frac{1-2i}{1-2i} e^{x+i2y} = R.P. \frac{1-2i}{5(1+4)} e^x e^{i2y} \\ &= R.P. \frac{1-2i}{25} e^x [\cos(2y) + i\sin(2y)] = \frac{e^x}{25} [\cos(2y) + 2\sin(2y)]. \end{aligned}$$

$z = f_1(y+x) + f_2(y-x) + x\ f_3(y-x) + \frac{e^x}{25}(\cos 2y + 2\sin\ 2y)$.

Example 30.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(2x + y)$.
Solution. *The complementary function is* $f_1(y - x) + x\ f_2(y - x) + f_3(y + x)$.

$$
\begin{aligned}
P.I &= \frac{1}{D^3 + D^2D' - DD'^2 - D'^3}\cos(2x+y) \\
&= R.P.\frac{1}{D^3 + D^2D' - DD'^2 - D'^3}e^{i(2x+y)} \\
&= R.P.\frac{1}{(2i)^3 + (2i)^2(i) - (2i)(i)^2 - (i)^3}e^{i(2x+y)} \\
&= R.P.\frac{1}{-8i - 4i + 2i + i}e^{i(2x+y)} \\
&= R.P.\frac{1}{-9i}e^{i(2x+y)} \\
&= R.P.\frac{i}{9}[\cos(2x+3y) + i\ \sin(2x+y)] \\
&= -\frac{1}{9}\sin(2x+y).
\end{aligned}
$$

$$z = f_1(y-x) + x\ f_2(y-x) + f_3(y+x) - \frac{1}{9}\sin(2x+y).$$

Example 31.

Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = \cos(x+y)$.

Solution.

$$\text{The auxillary equation is } m^3 + m^2 - m - 1 = 0$$
$$m^2(m+1) - (m+1) = 0$$
$$(m^2-1)(m+1) = 0$$
$$(m^2-1)(m+1) = 0$$
$$m = 1, -1, -1.$$
$$C.F = f_1(y+x) + f_2(y-x) + x\ f_3(y-x).$$

$$P.I = \frac{1}{D^3 + D^2D' - DD'^2 - D'^3}\cos(x+y) = R.P\frac{1}{(D-D')(D^2+2DD'+D'^2)}e^{i(x+y)}$$
$$= R.P.\frac{1}{((i)^2 + 2(i)(i) + (i)^2)}x\ e^{i(x+y)} = R.P.\frac{1}{(-1-2-1)}x\ e^{i(x+y)} = R.P.\frac{1}{-4}xe^{i(x+y)}$$
$$= R.P. -\frac{1}{4}x(\cos(x+y) + i\ \sin(x+y)) = -\frac{1}{4}\ x\cos(x+y).$$

$z = f_1(y+x) + f_2(y-x) + xf_3(y-x) - \frac{1}{4}\ x\cos(x+y).$

Case(iii).

If $G(x,y) = x^r y^s$, then the particular integral is given by

$$P.I = \frac{1}{F(D,D')} x^r y^s = [FD, D']^{-1} x^r y^s,$$

Now expand $[F(D,D')]^{-1}$ as a binomial series and operate on $x^r y^s$.

Example 32.

Solve $(D^2 - 2DD')z = x^3 y$.

Solution. *Complementary function is* $F = f_1(y) + f_2(y+2x)$.

$$P.I = \frac{1}{D^2 - 2DD'} x^3 y = \frac{1}{D^2\left[1 - \frac{2D'}{D}\right]} x^3 y = \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1} x^3 y$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{4D'^2}{D^2} + \cdots\right] x^3 y = \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{4{D'}^2}{D^2}\right]^{-1} x^3 y$$

$$= \frac{1}{D^2}\left[x^3 y + \frac{2}{D}x^3 + 0\right] = \frac{1}{D^2}\left[x^3 y + \frac{2x^4}{4} + 0\right] = \frac{x^5 y}{4 \times 5} + \frac{x^6}{2 \times 5 \times 6} = \frac{x^5 y}{20} + \frac{x^6}{60}.$$

$z = f_1(y) + f_2(y+2x) + \frac{x^5 y}{20} + \frac{x^6}{60}$.

Example 33.

Solve $(D^2 + 2DD' + D'^2)z = x^2 + xy - y^2$.
Solution. *The complementary function is* $f_1(y-x) + x\ f_2(y-x)$.

$$P.I = \frac{1}{D^2 + 2DD' + D'^2}(x^2 + xy - y^2) = \frac{1}{D^2\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]}(x^2 + xy - y^2)$$

$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right]^{-1} x^2 + xy - y^2$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D} - \frac{D'^2}{D^2} + \frac{4D'^2}{D^2} + \cdots\right] x^2 + xy - y^2$$

$$= \frac{1}{D^2}\left[x^2 + xy - y^2 - \frac{2}{D}(x - 2y) + 3\frac{1}{D^2}(-2)\right]$$

$$= \frac{1}{D^2}[x^2 + xy - y^2 - x^2 + 4xy - 3x^2]$$

$$= \frac{1}{D^2}[5xy - y^2 - 3x^2]$$

$$= \left[\frac{5}{6}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{4}x^4\right].$$

$$z = f_1(y-x) + x\ f_2(y-x) + \frac{5}{6}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{4}x^4.$$

Case (iv)

If $G(x,y) = e^{ax+by}x^r y^s$ or $\cos\ ax + by\ x^r y^s$ or $\sin ax + by\ x^r y^s$ the particular integral is given by

$$\begin{aligned} P.I. &= \frac{1}{F(D,D')}e^{(ax+by)}x^r y^s = \frac{e^{(ax+by)}}{F(D+a,D'+b)}x^r y^s \\ &= e^{(ax+by)}[F(D+a,D'+b)]^{-1}x^r y^s. \end{aligned}$$

Expand $[F(D+a.D'+b)]^{-1}$ as a binomial series and operate on $x^r y^s$.

$$\begin{aligned} P.I. &= \frac{1}{F(D,D')}\cos^{(ax+by)} x^r y^s = R.P.\frac{1}{F(D,D')}e^{i(ax+by)}x^r y^s \\ &= R.P.\frac{e^{i(ax+by)}}{F(D+ai,D'+bi)}x^r y^s \\ &= R.P.e^{i(ax+by)}[F(D+ai,D'+bi)]^{-1}x^r y^s. \end{aligned}$$

Expand $[F(D+ai,D'+bi)]^{-1}$ as a binomial series and operate on $x^r y^s$.

Case (iv)

$$P.I. = \frac{1}{F(D,D')}\sin(ax+by)x^r y^s =$$

$$I.P.\frac{1}{F(D,D')}e^{i(ax+by)}x^r y^s$$

$$= I.P.\frac{e^{i(ax+by)}}{F(D+ai,D'+bi)}x^r y^s$$

$$= I.P.e^{i(ax+by)}[F(D+ai,D'+bi)]^{-1}x^r y^s.$$

Expand $[F(D+ai,D'+bi)]^{-1}$ as a binomial series and operate on $x^r y^s$.

Example 34.

Solve $\frac{\partial z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial x^2} = y \cos x.$

Solution. *The complementary function is* $f_1(y+2x) + f_2(y-3x)$.

$$
\begin{aligned}
P.I &= \frac{1}{D^2 + DD' - 6D'^2} y \cos x = R.P. \frac{e^{ix}}{D^2 + DD' - 6D'^2} y \\
&= R.P. \frac{e^{ix}}{-1 + 2iD + D^2 + iD' + DD' - 6D'^2} y \\
&= R.P. \frac{e^{ix}}{-[1 - \{iD' + 2iD + D^2 + DD' - 6D'^2\}]} y \\
&= -R.P. e^{ix}[1 - (iD' + 2iD + D^2 + DD' - 6D'^2)]^{-1} y \\
&= -R.P. e^{ix}[1 - (iD' + 2iD + D^2 + DD' - 6D'^2)] y \\
&= -R.P. e^{ix}[y + iD'(y)] = -R.P. (\cos x + i \sin x)[y + i] \\
&= -y \cos x + \sin x \\
z &= f_1(y+2x) + f_2(y-3x) - y \cos x + \sin x.
\end{aligned}
$$

Example 35.

Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$.
Solution. *The complementary function is* $f_1(y+2x) + f_2(y-x)$.

$$
\begin{aligned}
P.I &= \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x \\
&= \frac{1}{D^2 - DD' - 2r^2}(y-1)e^x \\
&= \frac{e^x}{(D+1)^2 - (D+1)(D') - 2D'^2}(y-1) \\
&= \frac{e^x}{1 + 2D + D^2 - D'D - D' - 2D'^2}(y-1) \\
&= \frac{e^x}{[1 + (2D + D^2 - D' - DD' - 5D'^2)]}(y-1) \\
&= e^x[1 + (2D + D^2 - D' - DD' - 5D'^2)]^{-1}(y-1) \\
&= e^x[1 + (2D + D^2 - D' - DD' - 5D'^2)](y-1) \\
&= e^x[(y-1) + D'(y-1)] \\
&= e^x[y - 1 + 1] \\
&= ye^x. \\
z &= f_1(y+2x) + f_2(y-x) + ye^x.
\end{aligned}
$$

Example 36.

Solve $(D^2 - 5DD' + 6D'^2)z = y \sin\ x$.

Solution. *The complementary function is* $f_1(y+2x) + f_2(y+3x)$.

$$
\begin{aligned}
P.I. &= \frac{1}{D^2 - 5DD' + 6D'^2} y \ \sin\ x = I.P. \frac{1}{D^2 - 5DD' + 6D'^2} e^{ix} y \\
&= I.P. \frac{e^{ix}}{(D+i)^2 - 5(D+i)(D') - 6D'^2} y \\
&= I.P. \frac{e^{ix}}{-1 + 2id + D^2 - 5iD' - 5DD' - 6D'^2} y \\
&= I.P. \frac{e^{ix}}{-[1 + (5iD' - 2iD - D^2 + 5DD' + 6D'^2)]} y \\
&= I.P. - e^{ix}[1 + (5iD' - 2iD - D^2 + 5DD' + 6D'^2)]^{-1} y \\
&= I.P. - e^{ix}[1 - (5iD' - 2iD - D^2 + 5DD' + 6D'^2)] y \\
&= I.P. - e^{ix}[y - 5iD'(y)] = I.P. - (\cos x + i \sin x)[y - 5i] \\
&= 5 \cos x - y \sin x. \\
z &= f_1(y+2x) + f_2(y+3x) + 5 \cos x - y \sin x.
\end{aligned}
$$

Example 37.

1. *Solve* $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$
2. *Solve* $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$.
3. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3)z = e^{2x+y} + \cos(x + y)$.
4. *Solve* $(D^2 - 2DD')z = x^3y + e^{2x}$.
5. *Solve* $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x + 2y) + e^{2x+y}$.
6. *Solve* $(D^2 + 4DD' - 5D'^2)z = \sin(x - 2y) + 3e^{2x-y}$.
7. *Solve* $(D^2 - 6DD' + 5D'^2)z = e^x \sinh\ y + xy$.

Consider the equation of the form

$$(D - mD' - a)z = 0 \tag{1}$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$. Then (1) becomes $p - mq = az$ which is a Lagrange equation. Hence the subsidiary equation is

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}.$$

By taking the first two ratios, we get

$$y + mx = c_1. \tag{2}$$

By taking the first and third ratios, we have

$$\frac{dx}{1} = \frac{dz}{az} \implies \frac{z}{e^{ax}} = c_2. \tag{3}$$

The complete solution of equation (1) is given by

$$\frac{z}{e^a x} = f(y + mx) = e^{ax} f(y + mx).$$

Now we consider the general form of non homogeneous equation as

$$(D - m_1 D' - a_1)(D - m_2 D' - a_2) \cdots (D - m_n D' - a_n) z = 0$$

whose solution is given by

$$z = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \cdots + e^{a_n x} f_n(y + m_n x).$$

In the case of repeated-factors

$$(D - mD' - a)^r z = 0.$$

The solution is given by

$$z = e^{ax} f_1(y + mx) + x\ e^{ax} f_2(y + mx) + \cdots + x^{r-1} e^{ax}.$$

Example 38.

Solve $(D - 2D' - 3)(D - 3D' - 2)z = 0.$

Solution. *The given equation is* $(D - 2D' - 3)(D - 3D' - 2)z = 0$. *By comparing this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = 3, m_1 = 2$ *and* $m_2 = 3$.

$$z = e^{3x} f_1(y + 2x) + e^{2x} f_2(y + 3x).$$

Example 39.

Solve $(D^2 - DD' + D' - 1)z = 0.$

Solution. *The given equation is* $(D - D' + 1)(D - 1)z = 0$. *By comparing this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$ *Here* $a_1 = -1, a_2 = 1, m_1 = 1$ *and* $m_2 = 0$.

$$z = e^{-x} f_1(y + x) + e^{x} f_2(y).$$

Example 40.

Solve $(D^2 + 2DD' + D'^2 + 3D + 3D' + 2)z = e^{3x+5y}$.

Solution. *The given equation is* $(D + D' + 1)(D + D' + 2)z = 0$. *By comparing this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = -1, a_2 = -2, m_1 = -1$ *and* $m_2 = -1$.

$$C.F = e^{-x} f_1(y - x) + e^{-2x} f_2(y - x).$$

$$\begin{aligned} P.I &= \frac{1}{(D + D' + 1)(D + D' + 2)} e^{3x+5y} \\ &= \frac{1}{(3 + 5 + 1)(3 + 5 + 2)} e^{3x+5y} \\ &= \frac{1}{90} e^{3x+5y}. \end{aligned}$$

$$z = e^{-x} f_1(y - x) + e^{-2x} f_2(y - x) + \frac{1}{90} e^{3x+5y}.$$

Example 41.

Solve $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$.

Solution. *The given equation can be written as* $(D - D' - 1)(D - D' - 2)z = e^{6x} + 4e^{-4y} + 4e^{3x}e^{-2y}$. *To find C.F. compare this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = 1, a_2 = 2, m_1 = 1$ *and* $m_2 = 1$.

$$C.F = e^x f_1(y+x) + e^{2x} f_2(y+x).$$

$$\begin{aligned}
P.I &= \frac{1}{(D-D'-1)(D-D'-2)} e^{6x} + 4e^{-4y} + 4e^{3x-2y} \\
&= \frac{1}{(D-D'-1)(D-D'-2)} e^{6x} + \frac{1}{(D-D'-1)(D-D'-2)} 4e^{-4y} \\
&\qquad + \frac{1}{(D-D'-1)(D-D'-2)} 4e^{3x-2y} \\
&= \frac{1}{(6-1)(6-2)} e^{6x} + \frac{1}{(-(-4)-1)(-(-4)-2)} 4e^{-4y} + \frac{1}{(4)(3-(-2)-2)} 4e^{3x-2y}. \\
&= \frac{e^{6x}}{20} + \frac{e^{-4y}}{3} + \frac{e^{3x-2y}}{3}.
\end{aligned}$$

$$z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{e^{6x}}{20} + 2\frac{e^{-4y}}{3} + \frac{e^{3x-2y}}{3}.$$

Example 42.

Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$.

Solution. *The given equation can be written as* $(D + D')(D + D' - 2)z = \sin(x + 2y)$. *To find C.F. compare this equation with* $(D - m_1 D' - a_1)(D - m_2 D' - a_2)z = 0$. *Here* $a_1 = a, a_2 = 2, m_1 = -1$, *and* $m_2 = -1$.

$C.F. = f_1(y - x) + e^{2x} f_2(y - x)$

$$
\begin{aligned}
P.I &= \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \\
&= I.P. \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{i(x+2y)} \\
&= I.P. \frac{1}{i^2 + 2(i)(2i) + (2i)^2 - 2(i) - 2(2i)} e^{i(x+2y)} \\
&= I.P. \frac{1}{-1 - 4 - 4 - 2(i) - 2(2i)} e^{i(x+2y)} = I.P. - \frac{e^{i(x+2y)}}{3} \frac{1}{3 + 2(i)} \frac{3 - 2i}{3 - 2i} \\
&= I.P. - \frac{\cos(x + 2y) + i \sin(x + 2y)}{3} \frac{3 - 2i}{9 + 4} \\
&= \frac{1}{39}(2\cos(x + 2y) - 3\sin(x + 2y)).
\end{aligned}
$$

$$z = f_1(yx) + e^{2x} f_2(y - x) + \frac{1}{39}(2\cos(x + 2y) - 3\sin(x + 2y)).$$

References

T. Amaranath, *An Elementary Course in Partial Differential Equations*, Second Edition, Narosa Publishing House, 1997.

Ian Sneddon, *Elements of Partial Differential Equations*, McGraw-Hill Book Company, 1957.